Linear Algebra

[KOMS119602] - 2022/2023

14.2 Matrix Decomposition: SVD

sumber: *Slide Perkuliahan Aljabar Linear dan Geometri - R. Munir ITB*

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Learning objectives

After this lecture, you should be able to:

- explain the importance of singular value decomposition;
- perform singular value decomposition on a matrix.

Matrix decomposition

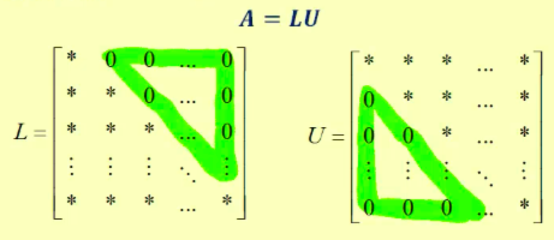
Decomposing matrix means factoring a matrix into **product of matrices**

Example: $A = P_1 \times P_2 \times \cdots \times P_k$

Matrix decomposition methods:

1. LU-decomposition (LU = Lower-Upper)
2. QR-decomposition (Q: othogonal, R: upper-triangular)
3. Singular value decomposition (SVD)

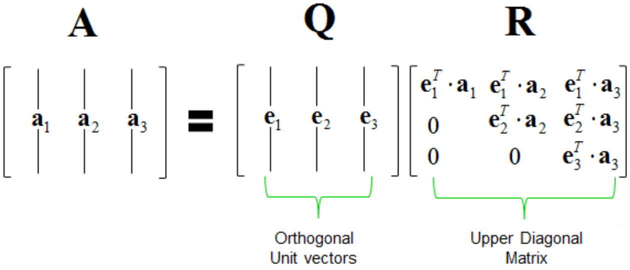
LU-decomposition



Example:

$$\begin{bmatrix} 2 & -3 & -1 \\ 3 & 2 & -5 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{1}{2} & \frac{11}{13} & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -1 \\ 0 & \frac{13}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{32}{13} \end{bmatrix}$$

QR-decomposition



Example:

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 2\sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

Singular Value Decomposition

Motivation of Singular Value Decomposition

In orthogonal diagonalization, a square $n \times n$ matrix A can be decomposed into:

$$A = P^T DP$$

dimana:

- P is an orthogonal matrix whose columns are eigenbases of A (so $P^T = P^{-1}$)

$$P = [p_1 \mid p_2 \mid \ldots \mid p_n]$$

- D is diagonal matrix, such that

$$D = P^{-1}AP$$

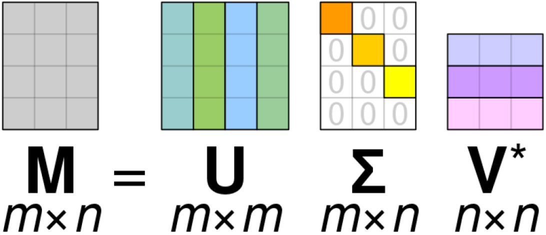
How to factorize non-square $m \times n$ matrix that do not have eigenvalues?

Singular Value Decomposition (SVD)

SVD is used to factorize non-square $m \times n$ matrix into product of matrix U, Σ, and V, such that:

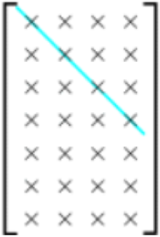
$$A = U\Sigma V^T$$

- U is an orthogonal $m \times m$ matrix
- V is an orthogonal $n \times n$ matrix
- Σ is an $m \times n$ matrix, whose elements in the *main diagonal* are singular values of A, and other elements are 0



Main diagonal of a non-square matrix

Main diagonal of a non-square matrix A of size $m \times n$, is defined as *entries* a_{11} *diagonally until* a_{mm} (assuming that $n > m$).



Orthogonal matrix (revisited)

Orthogonal matrix is a matrix whose columns form a set of orthogonal vectors. ($\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$).

If P is an orthogonal matrix, then $P^{-1} = P^T$.

Proof.

Vectors $v_1, v_2, \ldots, v_n$ of Q are orthogonal:

$$v_i^T v_j = \begin{cases} 0 & \text{if} i \neq j \\ 1 & \text{if} i = j \end{cases}$$

Let $P = [v_1 \mid v_2 \mid \ \ldots \ \mid v_n]$, then:

$$P^T \cdot P = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} [v_1 \mid v_2 \mid \cdots \mid v_n] = \begin{bmatrix} v_1^T v_1 & \cdots & v_1^T v_n \\ v_2^T v_1 & \cdots & v_2^T v_n \\ \cdots & \ddots & \cdots \\ v_n^T v_1 & \cdots & v_n^T v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Example

Are the following matrices orthogonal?

- $P = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$
- $Q = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

Singular values

Let A be an $m \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A^T A$, then:

$$\tau_1 = \sqrt{\lambda_1},\ \tau_2 = \sqrt{\lambda_2},\ \ldots, \tau_n = \sqrt{\lambda_n}$$

are called the singular values of A.

> *In this case, we assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, so that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq 0$.*

Theorem

Orthogonally diagonalizable matrices have positive eigenvalues If A is an $m \times n$ matrix, then:

1. *$A^T A$ is orthogonally diagonalizabe*
2. *The eigenvalues of $A^T A$ are non-negative*

Example

Given matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example

Given matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$$B = A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solve the characteristic equation:

$$\det(\lambda I - B) = 0 \iff \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = 0 \iff (\lambda - 2)(\lambda - 2) - 1 = 0$$

This gives $\lambda^2 - 4\lambda + 3 = 0 \iff (\lambda - 3)(\lambda - 1) = 0$
The eigenvalues pf AA^T are $\lambda_1 = 3$ and $\lambda_2 = 1$.

Hence:

$$\tau_1 = \sqrt{3} \quad \text{and} \quad \tau_2 = \sqrt{1} = 1$$

Decomposing matrix $A_{m \times n}$ into products of U, Σ, and V

1. Untuk vektor singular kiri, hitung nilai-nilai eigen dari AA^T.
2. Tentukan vektor-vektor eigen $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ yang berkoresponden dengan nilai-nilai eigen dari AA^T. Normalisasi vektor eigen-nya sehingga diperoleh:
$$U = \left[\frac{\mathbf{u}_1}{|\mathbf{u}_1|} \mid \frac{\mathbf{u}_2}{|\mathbf{u}_2|} \mid \ldots \mid \frac{\mathbf{u}_m}{|\mathbf{u}_m|} \right]$$
3. Untuk vektor singular kiri, hitung nilai-nilai eigen dari A^TA.
4. Tentukan vektor-vektor eigen $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ yang berkoresponden dengan nilai-nilai eigen dari A^TA. Normalisasi vektor eigen-nya sehingga diperoleh:
$$V = \left[\frac{\mathbf{v}_1}{|\mathbf{v}_1|} \mid \frac{\mathbf{v}_2}{|\mathbf{v}_2|} \mid \ldots \mid \frac{\mathbf{v}_m}{|\mathbf{v}_m|} \right]$$
5. Bentuklah matriks Σ berukuran *mxn* dengan elemen-elemen diagonalnya adalah *nilai-nilai singular* dari matriks A (yaitu $\tau_1 = \sqrt{\lambda_1},\ \tau_2 = \sqrt{\lambda_2},\ \ldots, \tau_n = \sqrt{\lambda_n}$) dari besar ke kecil.
6. Maka: $A = U\Sigma V^T$

Example

Given matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Find the Singular Value Decomposition of A.

Example

Given matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Find the Singular Value Decomposition of A.

For the solution, check the pdf file.

Applications of Singular Value Decomposition

- Image and video compression
- Image processing
- Machine learning
- Computer vision
- Digital watermarking
- ...?
- ...?